Abstract Algebra & Number Theory
Summer 2009

Text: *Algebra: Abstract and Concrete* by Frederick Goodman. Available online at

http://www.math.uiowa.edu/ goodman/algebrabook.dir/algebrabook.html

The development of the following understandings will be the primary goal of this course:

- Algebra is useful, universal and relevant in everyday life and in K-12 mathematics teaching.
- Algebra is a living organism, algebraic tools are developed to enable better understanding of the world around us, and the ability to think abstractly is a valuable asset.
- Algebraic properties, rules, and concepts make sense.
- Mathematics is a language communicated through specialized vocabulary and symbols used to represent and describe mathematical ideas, generalizations, and relationships.
- Symmetry plays a key role in science, math, engineering and culture.

Objectives:

- To explore a fascinating, beautiful new world of mathematics.
- To learn the fundamental properties of the basic algebraic structures (rings, fields, groups) from both a concrete and abstract point of view.
- To develop the ability to think abstractly, and to clearly explain difficult ideas and thoughts.

Expectations

- Work hard and regularly. This will be a difficult class. Re-write your notes.
- Own the fundamental definitions and main results.
- Create, and be familiar with a large inventory of examples. The best way to combat abstractness is to have some concrete examples to play with.
- Be willing to seek help. There will be times in this course that you will be frustrated. This is natural. This is not an easy course. Anytime you are learning a difficult new task, you encounter frustration. Don't let the frustration get you down. Come in and talk with me!
- **Enjoy!** You will be exploring an exciting new world, and will be challenging your mind. What a wonderful opportunity!

Grading

Projects	25%
Field Guide	25%
Capstone project	25%
Understandings Exams	25%

Projects

Throughout the course there will be projects, some in-class and some outside of class; some individual and some in groups. For some of these you will be assessed on your participation (are you actively asking questions? are you sharing your thought processes with others?). For others, the group or individual will hand-in a brief report which will be graded based upon mathematical correctness, completeness, creativity, and depth of understanding.

Field guide

Much of this class deals with new and abstract objects. To help better understand these objects, you will put together a *Field Guide of Abstract Algebra & Number Theory*. The field guide will contain material that helps you understand the new objects that we will encounter. Correct definitions, examples, illustrations, explanations and connections to secondary mathematics should be included. The field guide will be collected twice during the course, and graded on correctness, completeness, and richness.

Capstone Project

Two-thirds of the way through the course, groups of students will choose a topic (either from a list to be provided, or one agreed upon by the instructor) in Algebra & Number Theory to study in more depth. During the last week of class each group will present their topic to the class (either as a formal lecture, a power points presentation, a discovery-activity for the class, etc).

Understandings Exams

Towards the end of the course there will be a "take-home" exam that will allow you to ply your newly developed skills and understandings.

Important points from
The Mathematical Education of Teachers

Below we give some recommendations about the mathematical education of teachers from the 2001 Conference Board of the Mathematical Sciencesreport: *The Mathematical Education of Teachers.* The entire report is available online at

http://www.cbmsweb.org/MET_Document/index.htm

General Recommendations

- **Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach.**
  Prospective teachers need to understand the fundamental principles that underlie school mathematics, so that they can teach it to diverse groups of students as a coherent, reasoned activity and communicate an appreciation of the elegance and power of the subject. With such knowledge, they can foster an enthusiasm for mathematics and a deeper understanding among their students. Prospective teachers need to develop a thorough mastery of the mathematics in several grades beyond that which they expect to teach, as well as of the mathematics in earlier grades.

- **Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas.**
  Attention to the broad and flexible applicability of basic ideas and modes of reasoning is preferable to superficial coverage of many topics. Prospective teachers should learn mathematics in a coherent fashion that emphasizes the interconnections among theory, procedures, and applications. They should learn how basic mathematical ideas combine to form the framework on which specific mathematics lessons are built.

- **Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching.**
  Mathematics is not only about numbers and shapes, but also about patterns of all types. In searching for patterns, mathematical thinkers look for attributes like linearity, periodicity, continuity, randomness, and symmetry. They take actions like representing, experimenting, modeling, classifying, visualizing, computing, and proving. Teachers need to learn to ask good mathematical questions, as well as find solutions, and to look at problems from multiple points of view. Most of all, prospective teachers need to learn how to learn mathematics.

- **Teachers need the opportunity to develop their understanding of mathematics and its teaching throughout their careers, through**

both self-directed and collegial study, and through formal course-work.

Recommendations regarding Algebra & Number Theory
To be well-prepared to teach such high school curricula, mathematics teachers need:

- Understanding of the properties of the natural, integer, rational, real, and complex number systems.
- Understanding of the ways that basic ideas of number theory and algebraic structures underlie rules for operations on expressions, equations, and inequalities.
- Understanding and skill in using algebra to model and reason about real-world situations.
- Ability to use algebraic reasoning effectively for problem solving and proof in number theory, geometry, discrete mathematics, and statistics.
- Understanding of ways to use graphing calculators, computer algebra systems, and spreadsheets to explore algebraic ideas and algebraic representations of information, and in solving problems.

Courses in abstract algebra and number theory examine mathematical structures that are the foundation for number systems and algebraic operations. These courses should assure that future teachers "know why" the number systems and algebra operate as they do. Unfortunately, too many prospective high school teachers fail to understand connections between those advanced courses and the topics of school algebra.

Prospective teachers can be helped to make these connections. Courses can be infused with tasks that ask for specific instances of these connections, for example, to show explicitly how the number and algebra operations of secondary school can be explained by more general principles. Assignments might ask for the use of unique factorization and the Euclidean algorithm to justify familiar procedures for finding common multiples and common divisors of integers and polynomials, or for the solution of a linear or quadratic equation to be justified by a field property; and to show how each extension of the number system, from natural numbers through complex numbers, is accompanied by new properties. Each facet in such courses would provide teachers with insight into the structure of high school mathematics, its uses in science and technology or in the workplace, and the conceptual difficulties in learning number and algebraic concepts.

A course in Algebra & Number theory should develop students' appreciation of the breadth and power of algebraic structures.

Mathematical maturity and attitudes that will enable and encourage continued growth of knowledge in the subject and its teaching.

There are at least five themes that could be addressed in a capstone look at algebra and number theory.

- **Historical perspectives** Prospective teachers need to develop an eye for the ideas of mathematics that will be particularly challenging for their students. One very useful guide to such topics is the historical record showing how the ideas were first developed. The history of algebra is rich in stories that illuminate the challenge of expressing number patterns and properties in modern notation, of developing algorithms for solution of equations, of making sense of fractions, negative numbers, irrational numbers, and complex numbers.

  Investigation of these historical issues involves substantial mathematics. It also provides prospective high school mathematics teachers with insight for teaching that they are unlikely to acquire in courses for mathematics majors headed to graduate school or technical work.

- **Common misconceptions** Although historical analysis shows the difficulties encountered in development of fundamental algebraic ideas and techniques, there are many aspects of algebra that seem to be persistent challenges in learning the subject. Experienced teachers come to know where students are likely to make mistakes in algebraic manipulation and reasoning. It makes sense to address these issues in the preparation of high school teachers and to analyze the mathematical and psychological factors that lead students to common errors.

- **Applications**
  Because algebra occurs throughout all branches of mathematics, it is easy for students and prospective teachers to get the impression that it is a tool in the service of other topics. Many applications of algebra do occur in the context of problems in geometry or analysis, but there are some very important applications that apply core algebraic topics directly. For example, linear programming problems make an excellent setting for illustrating the usefulness of core algebra topics like linear equations and inequalities. Cryptography is a direct application of both abstract algebra and number theory. Aspects of both linear programming and cryptography can be developed at levels that are appropriate for and interesting to high school algebra students.

- **Technology**
  College algebra and number theory courses are beginning to exploit and study the use of calculator and computer tools for doing and learning algebraic ideas. Calculators with powerful computer algebra systems (CAS) are now available at prices that make them accessible, and they are used

more and more in schools. With only modest instruction, a high school student can use such tools to do most of the calculation problems that fill course assignments and examinations in algebra. The implications of this new-found calculating power are still being worked out in a variety of formal and informal classroom and curriculum experiments. High school teachers whose careers will cover the next 30 or 40 years need experience with these tools in their own learning and problem solving that will usefully inform their teaching. There is growing evidence that use of CAS technology can expand high school algebra and that intelligent users need common sense and mathematical habits of mind to use these tools wisely.

In addition to CAS software, computers offer other tools that require algebraic understanding. For example, spreadsheets are one of the most widely used computer tools, and designing a useful spreadsheet requires flexible ability to express numerical relationships in algebraic notation. College mathematics courses not often incorporate spreadsheet tasks. However, high school computer literacy courses often introduce students to this tool, and there are important ways that spreadsheet work can illuminate important mathematical ideas. For example, spreadsheet formulas often implicitly define recursive procedures, so spreadsheets can be used to explore fundamental properties of arithmetic, geometric, and more general sequences and series.

- **Connections**
  Because algebra is the language in which so many relationships are expressed throughout mathematics, it is a natural topic to use in exploring important mathematical connections.